

Divisible quantum dynamics satisfies temporal Tsirelson's bound

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We show that divisibility of qubit quantum processes implies temporal Tsirelson's bound. We also prove that the classical bound of the temporal Bell's inequality holds for dynamics that can be described by entanglement-breaking channels—a more general class of dynamics than that allowed by classical physics.

Introduction.—Two classical systems interrogated by space-like separated measurements give rise to correlations bounded by Bell's inequalities [1]. Remarkably, quantum systems can violate such bounds, although they cannot achieve the maximal algebraically allowed value [2]. The quantum maximum, dubbed Tsirelson's bound [3], stems from reasons that are now well understood: violation of this bound would trivialise communication complexity [4, 5] and be against a number of natural postulates [6–12]. In a different yet related context, a number of works have studied correlations between the outcomes of time-like separated observables [13–23]. In this scenario, the reasons behind the existence of a Tsirelson-like bound, limiting the value taken by suitably built functions of two-time correlators, are not as clear. In this Letter we shed light on this fundamental question, showing that Tsirelson's bound for temporal correlations follows from a well-known and prevalent property of dynamical processes, namely their *divisibility* (see Ref. [24] for related first investigations).

Divisibility asserts that dynamical evolution between any two points in time can be decomposed into a series of intermediate-time evolutions. Fundamental dynamics is expected to be divisible and, indeed, the Schrödinger equation generates unitary processes, which are fully divisible. Furthermore, divisible evolutions are often good approximations to open-system dynamics and divisibility is assumed explicitly in the derivation of several master equations [25]. In fact, whenever the Markov assumption holds, i.e., the evolving system is memory-less, the process is divisible. This provides an intuition as to why divisibility might be the relevant feature for temporal Bell's inequalities. It is known that, in the temporal setting, both classical and quantum bounds on the temporal Bell's inequalities can be violated even with purely classical systems if they embody sufficiently large memory [23, 26], thus effectively breaking the divisibility condition. An explicit example of this will be given later on in this paper.

We also show that the usual “classical” bound on the temporal Bell's inequality holds for a more general class of dynamics than stochastic maps consistent with classical physics. This parallels the situation for space-like measurements, where the classical bound on Bell's inequality holds for local hidden variable models. These are strictly richer than classical ones, as illustrated for example in Ref. [27], where imposing invariance of measured correlations under rotations

of local coordinate systems is shown to lead to a more stringent version of Bell's theorem.

Scenario.— Consider the situation depicted in Fig. 1, where two observers, Alice and Bob, make time-ordered measurements with Alice measuring before Bob. Each choose to measure at one of two times; Alice (Bob) measures either at time t_1 or t_2 (t_3 or t_4). We allow for intermediate dynamics between any consecutive measurement times, and label the corresponding general quantum channels as Λ_A (Λ_B) for the evolution between t_1 and t_2 (t_3 and t_4), and Λ_E for the dynamics between t_2 and t_3 (c.f. Fig. 1). From their measurement outcomes the following temporal Bell function is constructed

$$\mathcal{B} = E_{13} + E_{14} + E_{23} - E_{24}. \quad (1)$$

Throughout this Letter we will be calculating the temporal Bell function above with various assumptions and restriction placed on Λ_A , Λ_E , and Λ_B .

Above, E_{ij} is the correlation function between the i th measurement performed by Alice and the j th one by Bob, with i, j denoting the instant of time at which measurements are performed, i.e., $i \in \{1, 2\}$ and $j \in \{3, 4\}$, which we call *time steps*. The correlation functions are defined as the expectation value of the product of measurement results obtained by Alice and Bob. They are calculated under the assumption that every experimental run is an independent event, i.e., without allow-

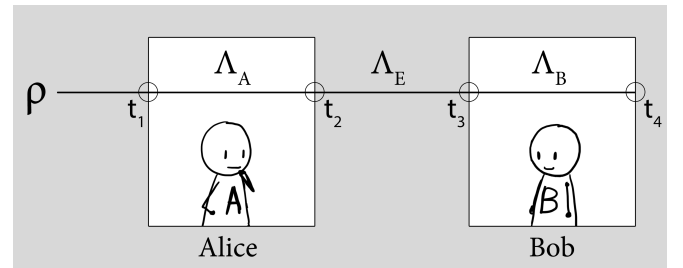


FIG. 1. Generalised temporal Clauser-Horne-Shimony-Holt scenario. The Λ_i are arbitrary channels (completely positive trace preserving maps). A physical system in a state ρ is first measured by Alice either at time t_1 or t_2 , and then by Bob either at time t_3 or t_4 . We show that the correlations observed by Alice and Bob, i.e., expectation values of the product of their results, satisfy temporal Tsirelson's bound whenever the Λ s are independent and hence the dynamics is divisible (see Def. 1).

ing for adaptive strategies where the measurement choices in a given run would depend on the outcomes obtained in previous experimental runs [28]. We consider dichotomic ± 1 observables parametrised by their corresponding Bloch vectors $\vec{\sigma} \cdot \vec{a}_i$ and $\vec{\sigma} \cdot \vec{b}_j$ of Alice and Bob respectively. The initial state is parametrised by $\rho = \frac{1}{2}(\mathbb{I} + \vec{\sigma} \cdot \vec{v})$.

Note that our model generalises that of Ref. [14] and reduces to their model when there is no dynamics between the two measurement choices of Alice and Bob. In this case, the temporal correlations can be turned into spatial correlations using the Choi-Jamiołkowski isomorphism; one can therefore resort to standard tools to recover Tsirelson's bound. However, in our scenario the correlation functions for different settings are measured on different states—a consequence of the different channels that act between each pair of measurement times. Our scenario is therefore richer, despite the fact that Alice and Bob perform the same number of measurements. For instance, Tsirelson's bound cannot be violated in the model of Ref. [14] (see App. C), while it can be in our model (see. Prop. 4). We show in this Letter that it is the divisibility of the process that enforces Tsirelson's bound for our model.

Definition 1. A process is N -divisible with respect to a set of times $\{t_1, t_2, \dots, t_N\}$ when the maps relating the system state between any two time-steps can be described by a composition of completely positive trace-preserving (CPTP) maps between intermediate times: $\Lambda_{l;j} = \Lambda_{l;k} \circ \Lambda_{k;j} \forall j, k, l$ where $t_l > t_k > t_j$.

Note that by defining $\Lambda_A \equiv \Lambda_{t_2;t_1}$, $\Lambda_B \equiv \Lambda_{t_4;t_3}$, and $\Lambda_E \equiv \Lambda_{t_3;t_2}$ independently, we are automatically imposing divisible dynamics with respect to $\{t_1, t_2, t_3, t_4\}$. Conversely, the process is indivisible when either Λ_E or Λ_B depend on Alice's measurement choice, as will be shown below. We will first study the classical bound of Eq. (1) to reveal that it is satisfied if Λ_E is an entanglement-breaking channel. We then prove that divisible dynamics leads to the temporal Tsirelson's bound. Finally, we study indivisible dynamics and its consequences on the temporal Bell's inequality.

Entanglement-breaking dynamics and classicality.—We begin by choosing the channel Λ_E to be any entanglement-breaking channel. Given an arbitrary entangled state of a composite system, a channel is entanglement-breaking and trace-preserving (EBT) if and only if its action on a subsystem yields a separable state. We shall prove that, in this case and for $\vec{v} = 0$ (i.e., the initial state completely mixed), $\mathcal{B}_{EBT} \leq \mathcal{B}_{cl} = 2$, so that we retrieve the well-known classical bound. Entanglement breaking channels include as a subset all stochastic processes. Thus, classicality for temporal correlations is proven for a larger class of evolutions.

We first show, by construction, that the assumption of $\vec{v} = 0$ is necessary, as relaxing it allows the Bell function in Eq. (1) to exceed the classical bound. Take Λ_A to be an identity channel, Λ_E to be a projective measurement along vector \vec{c} (clearly an EBT channel), and Λ_B to always output state \vec{b} independently of the input state. Further assume that Bob's measurements are $\vec{b}_1 = \vec{c}$ and $\vec{b}_2 = \vec{b}$. Then one easily verifies that the tem-

poral correlations read: $E_{i3} = \vec{a}_i \cdot \vec{c}$ and $E_{i4} = \vec{a}_i \cdot \vec{v}$. With this at hand, the Bell function in Eq. (1), which we label here \mathcal{B}_{EBT} , reaches its maximum if $\vec{a}_1 + \vec{a}_2$ is parallel to \vec{c} and $\vec{a}_1 - \vec{a}_2$ is parallel to \vec{v} , thus leading to $\mathcal{B}_{EBT} = 2\sqrt{1 + |\vec{v}|^2}$. The classical bound is thus violated for all input states with $|\vec{v}| > 0$. We now prove that if $|\vec{v}| = 0$, Eq. (1) satisfies the classical bound independently of the entanglement-breaking nature of the channel.

Theorem 2. If Λ_A and Λ_B are arbitrary CPTP channels, Λ_E is EBT, and $\vec{v} = 0$, then $\mathcal{B}_{EBT} \leq 2$.

Proof. Any CPTP map acting on a two-level system can be parameterized as $\Lambda(\vec{r}) = \vec{L} + \lambda \vec{r}$, where \vec{L} is a three-dimensional vector and λ is a matrix. We take $\vec{L} = \vec{A}$ and $\lambda = \alpha$ ($\vec{L} = \vec{B}$ and $\lambda = \beta$) for $\Lambda = \Lambda_A$ ($\Lambda = \Lambda_B$). In the following, we will make use of the following properties of EBT channels: *i)* EBT channels form a convex set; *ii)* for two-level systems, the extremal points of EBT channels are extremal classical-quantum (extremal CQ) maps (cf. Theorem 5D in Ref. [29]). Such extremal channels are one-dimensional projective measurements, and they output pure states. By Theorem 6 of Ref. [29], an extremal EBT channel acting on a qubit is fully specified using only two Kraus operators. Hence, if Λ_E is an extremal CQ channel, it can equivalently be thought of as a projective measurement along \vec{c} and a preparation of states with Bloch vectors \vec{r}_+ and \vec{r}_- .

The resulting correlation functions are listed in Appendix A and lead to the Bell function

$$\begin{aligned} \mathcal{B}_{EBT} = \frac{1}{2} \bigg[& (\vec{a}_2 \cdot \vec{c})(\vec{b}_1 \cdot \vec{s}) - (\vec{c} \cdot \vec{a}_2)(\vec{b}_2 \cdot \beta \vec{s}) \\ & + (\vec{a}_2 \cdot \vec{A})(\vec{b}_1 \cdot \vec{r} - 2\vec{b}_2 \cdot \vec{B} - \vec{b}_2 \cdot \beta \vec{r}) \\ & + (\vec{c} \cdot \alpha \vec{a}_1)(\vec{b}_1 \cdot \vec{s}) + (\vec{c} \cdot \alpha \vec{a}_1)(\vec{b}_2 \cdot \beta \vec{s}) \bigg], \quad (2) \end{aligned}$$

where $\vec{s} = \vec{r}_+ - \vec{r}_-$ and $\vec{r} = \vec{r}_+ + \vec{r}_-$. The Bell function is smooth in this case, hence numerical optimisation is robust. A constrained numerical optimization shows that $\max \mathcal{B}_{EBT} = 2$ [30]. By convexity, the proof holds when Λ_E is an arbitrary EBT channel.

Although a general analytical proof of the classical upper bound of Eq. (2) is non-trivial, the fairly minor restriction to only unital maps between Alice's time steps ($\vec{A} = 0$) leads to a straightforward proof. In this case, the first term in the second line of Eq. (2) vanishes and the classical upper bound can be verified analytically by noting that $\vec{b}_2 \cdot (\beta \vec{s}) = (\beta^T \vec{b}_2) \cdot \vec{s}$. \square

We have recovered the classical bound for the Bell function using a quantum system initially prepared in a maximally mixed state, where an EBT channel is placed between Alice's and Bob's measurements. The presence of such a channel between Alice and Bob in the evolution of the system is crucial: its absence would in general allow for the attainment of the quantum bound.

Suppose Λ_A is an extremal CQ channel with projective measurements along \vec{c} and output states with Bloch vectors

\vec{r}_+ and \vec{r}_- . If the input state is maximally mixed (i.e., $\vec{v} = 0$), the Bell function reads

$$\mathcal{B}_{EBT} = \frac{1}{2}(\vec{c} \cdot \vec{a}_1)(\vec{b}_1 + \vec{b}_2) \cdot (\vec{r}_+ - \vec{r}_-) + \vec{a}_2 \cdot (\vec{b}_1 - \vec{b}_2). \quad (3)$$

By choosing $\vec{b}_1 + \vec{b}_2 = 2 \cos \theta \vec{w}$ and $\vec{b}_1 - \vec{b}_2 = 2 \sin \theta \vec{w}_\perp$ with \vec{w} and \vec{w}_\perp two orthonormal vectors, one directly verifies that \mathcal{B}_{EBT} can achieve the temporal Tsirelson's bound.

On the other hand, we can still enforce $\mathcal{B} \leq 2$ even when the channel between Alice and Bob is entanglement-preserving. Let Λ_A and Λ_B both be identity channels, and consider the qubit channel of Werner type

$$\Phi_W = pI + (1-p)\frac{1}{2}\mathbb{1}, \quad p \in [0, 1], \quad (4)$$

where I is the identity channel and $\frac{1}{2}\mathbb{1}$ is the maximally incoherent channel, that replaces any input state with the maximally mixed one. The channel Φ_W is EBT if $p < 1/3$. Taking $\Lambda_E = \Phi_W$, the Bell function is $\mathcal{B}_W = p\mathcal{B}_1$, where $\mathcal{B}_1 \leq 2\sqrt{2}$ is the Bell function corresponding to the identity channel, i.e., for $p = 1$. For $1/3 < p < 1/\sqrt{2}$ the channel Φ_W is entanglement-preserving, yet $\mathcal{B}_W < 2$.

We have now shown the conditions which guarantee the classical bound $\mathcal{B}_{cl} = 2$, namely that the channel connecting Alice and Bob is entanglement breaking and the initial state is completely mixed. We have also shown that the ‘converse’ statement does not hold, i.e., having an entanglement-preserving channel between Alice and Bob and $|\vec{v}| > 0$ does not guarantee a violation of the classical bound. We now move to considerations of general quantum channels.

Temporal Tsirelson's bound for divisible processes.—We show that for generic CPTP maps, the temporal Bell function still obeys the quantum upper bound $\mathcal{B}_q = 2\sqrt{2}$. This is shown through the following.

Theorem 3. *For Λ_A , Λ_B and Λ_E arbitrary CPTP channels, we have $\mathcal{B} \leq 2\sqrt{2}$.*

Proof. We use the same sort of parameterization for $\Lambda_{A,B,E}$ introduced in the proof of Theorem 2. The input state is specified by the Bloch vector \vec{v} . We introduce new vectors $\vec{\xi}_1 = (\vec{v} \cdot \vec{a}_1)\vec{E} + \gamma[(\vec{v} \cdot \vec{a}_1)\vec{A} + \alpha\vec{a}_1]$ and $\vec{\xi}_2 = \gamma\vec{a}_2 + [(\vec{A} + \alpha\vec{v}) \cdot \vec{a}_2]\vec{E}$. In Appendix B it is shown that the Bell function can then be rewritten as

$$\mathcal{B}_q = \vec{\xi}_1 \cdot \vec{b}_1 + \vec{\xi}_2 \cdot \vec{b}_1 + (\beta\vec{\xi}_1) \cdot \vec{b}_2 - (\beta\vec{\xi}_2) \cdot \vec{b}_2 + [\vec{v} \cdot \vec{a}_1 - (\vec{A} + \alpha\vec{v}) \cdot \vec{a}_2](\vec{B} \cdot \vec{b}_2). \quad (5)$$

Note that for any channel $\Lambda(\vec{r}) = \vec{L} + \lambda\vec{r}$, we have $|f\vec{L} + \lambda\vec{r}| \leq 1$, with f a scalar such that $|f| \leq 1$. Therefore, both our new vectors $\vec{\xi}_{1,2}$ have at most unit length. If we now impose that channel Λ_B is unital, i.e., $\vec{B} = 0$, the second line of Eq. (5) vanishes and \mathcal{B}_q takes the form of the usual Bell function. Hence $\mathcal{B}_q \leq 2\sqrt{2}$. We have verified numerically that the same bound holds for a non-zero \vec{B} . Appendix C also shows that, if Λ_A ,

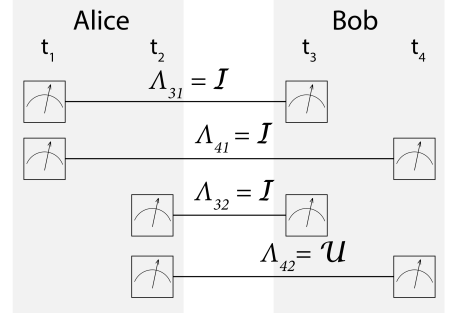


FIG. 2. Example of an indivisible process. Alice measures either at t_1 or t_2 , and Bob measures either at t_3 or t_4 . If Alice measures at t_2 , then there is a unitary channel in between $\{t_3, t_4\}$. All other channels are identity channels I . See main text for a proof that this dynamics is not divisible and leads to the maximal algebraic value of the temporal Bell function.

Λ_B are unitary and Λ_E is an arbitrary CPTP map, the temporal Tsirelson's bound is achieved only when Λ_E is a unitary map. \square

The proofs of both of our main Theorems are partially numerical. We point out that fully analytical arguments would be cumbersome. They are possible in the spatial scenario owing to the convexity of the set of states, which reduces the problem to a proof using pure states only. Furthermore, the use of the Schmidt decomposition reduces the number of variables over which one has to optimise. In the temporal case, on the other hand, a reduction in the number of variables is less forthcoming. Even if we consider only extremal maps, parameter counting shows that there are 30 variables to be optimised [c.f. Appendix D]. As such extremal CPTP maps are not unitary, the problem is clearly non-trivial.

We have shown that any divisible quantum process has its corresponding Bell function bounded from above by Tsirelson's bound. For unitary dynamics this was effectively shown in Refs. [14] and [17], but now it is clear that the relevant property of unitary transformation is their divisibility. Furthermore, non-unitary dynamics can also lead to correlations that achieve Tsirelson's bound as long as the input state is biased, i.e., $|\vec{v}| > 0$.

Indivisible processes.— A process is indivisible when it does not satisfy Definition 1. Such processes can be characterised by CPTP transformations using the superchannel formalism [31, 32].

Proposition 4. *Indivisible processes can yield $\mathcal{B}_{ID} > 2\sqrt{2}$.*

Proof. In Fig. 2, the channels Λ_{31} , Λ_{41} and Λ_{32} are identity channels and the last channel $\Lambda_{42} = \mathcal{U}$ is unitary. As the action of the unitary channel on a state is to rotate its Bloch vector, let R_u be the equivalent rotation of \mathcal{U} . The Bell function is thus

$$\mathcal{B}_q = (\vec{a}_1 + \vec{a}_2) \cdot \vec{b}_1 + [\vec{a}_1 - (R_u \vec{a}_2)] \cdot \vec{b}_2. \quad (6)$$

	No superposition	Superposition
Classical divisible	2	2 (EBT)
Quantum divisible	2	$2\sqrt{2}$
Indivisible	4	4

TABLE I. Summary of the results. The type of channel between Alice and Bob is presented in rows. Divisible classical processes are composed of stochastic maps in a fixed basis, whereas divisible quantum processes are described in Def. 1. If these maps act on states that are diagonal in a fixed basis (No superposition column) the correlations they allow for of course satisfy the classical bound. We show that the same bound is satisfied even if superpositions are allowed, i.e., for entanglement-breaking channels (EBT) between Alice and Bob. Composition of quantum maps gives quantum divisible processes, which include unitary dynamics. These processes can at most lead to Tsirelson’s bound for the temporal Bell function. Finally, indivisible processes can achieve the algebraic maximum of the Bell function.

By letting $\vec{a}_1 = \vec{a}_2 = \vec{b}_1 = \vec{b}_2$ and the unitary transformation \mathcal{U} be such that $R_u \vec{a}_2 = -\vec{b}_2 = -\vec{a}_2$, we get $\mathcal{B}_q = 4$, i.e., we violate Tsirelson’s bound and achieve the maximum algebraic value.

We now prove, by contradiction, that this process is indivisible. As $\Lambda_{32} = I$ and $\Lambda_{42} = \mathcal{U}$, divisibility would imply that $\Lambda_{43} = \mathcal{U}$. Divisibility of $\Lambda_{43} = \mathcal{U}$ and $\Lambda_{41} = I$ then imply that $\Lambda_{31} = \mathcal{U}^\dagger$. This contradicts the original assumption $\Lambda_{31} = I$. The process is thus indivisible. \square

In addition to the above, note that such an indivisible process can also be realised on a classical system. As all the Bloch vectors are either parallel or antiparallel, they encode classical information only. On the other hand, indivisibility is not sufficient to exceed the classical bound $\mathcal{B}_{cl} = 2$. If all the maps are the maximally incoherent map $\mathbb{1}$, which always outputs $\mathbb{1}/2$, then $B_{\mathbb{1}} = 0$. Denote the maps of the scenario in Proposition 4 as $\{I, \mathcal{U}\}$. If we take the convex combination $p\{I, \mathcal{U}\} + (1-p)\mathbb{1}$ then $B = pB_{\{I, \mathcal{U}\}} + (1-p)B_{\mathbb{1}} = pB_{\{I, \mathcal{U}\}} \leq 4p < 2$ for suitable choices of p , yet this remains indivisible.

Conclusions.— Contrary to the spatial Bell scenario, it is no longer possible to derive a non-trivial bound on the temporal Bell’s inequalities which would be independent of the physical systems themselves. This universality of the original Tsirelson’s bound is a consequence of essentially static spatial setting, i.e., the particles are only prepared and measured. Nevertheless we showed here a simple condition on the evolving physical system which guarantees that the temporal Tsirelson’s bound is satisfied. For the bound to hold the dynamics has to be divisible. Channel divisibility therefore plays a role for correlations in time similar to that played by information causality, macroscopic locality, etc. in space-like scenarios [6–12].

An interesting consideration is that a unitary process looks like an indivisible one from the classical perspective [33]. For instance, consider a three time-step process, where a measurement over the basis of eigenstates of σ_z is made at any two

time steps and between each time step the Hadamard gate is applied. The correlation functions involving the middle time step always vanish, while the correlation function between the initial and final time steps is 1. At the level of measurement outcomes, the dynamics involving the middle time step are described by fully noisy maps, while the evolution between the initial and final time steps is described by the identity channel. Therefore, from the classical perspective the channel is indivisible, while from the quantum perspective it is perfectly divisible. One can thus conjecture that such a distinction is responsible for the violation of the classical bound of temporal Bell’s inequalities and for reaching Tsirelson’s bound with unitary processes.

Finally, note that the indivisible process in Fig. 2 is no-signalling if the input state is maximally mixed, i.e., the outcomes of Bob do not reveal any information about the settings of Alice. Hence, it is not the ‘signalling’ that maximises the Bell function, but rather it is the non-Markovian memory of the process. In fact, we never imposed a no-signalling condition even for divisible processes; unlike in space-like correlated systems, time-like processes can of course carry information forward (from Alice to Bob).

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Appendix A: Explicit correlation functions when Λ_E is EBT (Part of Theorem 2)

Let Λ_A and Λ_B be CPTP and Λ_E be EBT. We parametrise them as follows:

$$\Lambda_A \left(\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{r}) \right) = \frac{1}{2} \left(\mathbb{1} + \vec{\sigma} \cdot (\vec{A} + \alpha \vec{r}) \right), \quad (\text{A1})$$

$$\Lambda_B \left(\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{r}) \right) = \frac{1}{2} \left(\mathbb{1} + \vec{\sigma} \cdot (\vec{B} + \beta \vec{r}) \right), \quad (\text{A2})$$

$$\Lambda_E(\rho) = \sum_{m=\pm 1} R_m \text{tr}[P_{\vec{c}}^m \rho], \quad (\text{A3})$$

where \vec{A} , and \vec{B} are vectors, and α and β are matrices such that $|\vec{A} + \alpha \vec{v}| \leq 1$ etc for any $|\vec{v}| \leq 1$. The pure states R_m have Bloch vectors \vec{r}_{\pm} for $m = \pm 1$. Furthermore, denote by $P_{\vec{a}_i}^k = \frac{1}{2}(\mathbb{1} + k\vec{\sigma} \cdot \vec{a}_i)$ the post-measurement state of Alice if she obtains result $k = \pm$ at time t_i . Similarly, $P_{\vec{b}_j}^l = \frac{1}{2}(\mathbb{1} + l\vec{\sigma} \cdot \vec{b}_j)$ denotes the post-measurement state of Bob if he observes outcome $l = \pm 1$ at time t_j . With this notation the correlation functions read:

$$\begin{aligned} E_{13} &= \sum_{k,\ell=\pm 1} k \cdot \ell \cdot \text{tr}[\rho P_{\vec{a}_1}^k] \text{tr}[\Lambda_{EBT}(\Lambda_A(P_{\vec{a}_1}^k)) P_{\vec{b}_1}^\ell] \\ &= \frac{1}{2} \left\{ (\vec{c} \cdot \alpha \vec{a}_1) \vec{b}_1 \cdot (\vec{r}_+ - \vec{r}_-) + (\vec{v} \cdot \vec{a}_1) \vec{b}_1 \cdot (\vec{r}_+ + \vec{r}_-) + (\vec{v} \cdot \vec{a}_1) (\vec{c} \cdot \vec{A}) \vec{b}_1 \cdot (\vec{r}_+ - \vec{r}_-) \right\} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} E_{14} &= \sum_{k,\ell=\pm 1} k \cdot \ell \cdot \text{tr}[\rho P_{\vec{a}_1}^k] \text{tr}[\Lambda_B(\Lambda_{EBT}(\Lambda_A(P_{\vec{a}_1}^k))) P_{\vec{b}_2}^\ell] \\ &= \frac{1}{2} (\vec{c} \cdot \alpha \vec{a}_1) \vec{b}_2 \cdot \beta (\vec{r}_+ - \vec{r}_-) + (\vec{v} \cdot \vec{a}_1) (\vec{b}_2 \cdot \vec{B}) + \frac{1}{2} (\vec{v} \cdot \vec{a}_1) \vec{b}_2 \cdot \beta (\vec{r}_+ (1 + \vec{c} \cdot \vec{A}) + \vec{r}_- (1 - \vec{c} \cdot \vec{A})) \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} E_{23} &= \sum_{k,\ell=\pm 1} k \cdot \ell \cdot \text{tr}[\Lambda_A(\rho) P_{\vec{a}_2}^k] \text{tr}[\Lambda_{EBT}(P_{\vec{a}_2}^k) P_{\vec{b}_1}^\ell] \\ &= \frac{1}{2} \left\{ (\vec{a}_2 \cdot \vec{c}) \vec{b}_1 \cdot (\vec{r}_+ - \vec{r}_-) + (\vec{a}_2 \cdot \vec{w}) \vec{b}_1 \cdot (\vec{r}_+ + \vec{r}_-) \right\} \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} E_{24} &= \sum_{k,\ell=\pm 1} k \cdot \ell \cdot \text{tr}[\Lambda_A(\rho) P_{\vec{a}_2}^k] \text{tr}[\Lambda_B(\Lambda_{EBT}(P_{\vec{a}_2}^k)) P_{\vec{b}_2}^\ell] \\ &= \frac{1}{2} (\vec{c} \cdot \vec{a}_2) \vec{b}_2 \cdot \beta (\vec{r}_+ - \vec{r}_-) + (\vec{a}_2 \cdot \vec{w}) (\vec{b}_2 \cdot \vec{B}) + \frac{1}{2} (\vec{a}_2 \cdot \vec{w}) \vec{b}_2 \cdot \beta (\vec{r}_+ + \vec{r}_-), \end{aligned} \quad (\text{A7})$$

where $\vec{w} = \vec{A} + \alpha \vec{v}$.

Appendix B: Derivation of (5) in Theorem 3

Let us parameterise arbitrary CPTP maps, Λ_A , Λ_E and Λ_B as

$$\Lambda_A \left(\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{r}) \right) = \frac{1}{2} \left(\mathbb{1} + \vec{\sigma} \cdot (\vec{A} + \alpha \vec{r}) \right) \quad (\text{B1})$$

$$\Lambda_E \left(\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{r}) \right) = \frac{1}{2} \left(\mathbb{1} + \vec{\sigma} \cdot (\vec{E} + \gamma \vec{r}) \right) \quad (\text{B2})$$

$$\Lambda_B \left(\frac{1}{2} (\mathbb{1} + \vec{\sigma} \cdot \vec{r}) \right) = \frac{1}{2} \left(\mathbb{1} + \vec{\sigma} \cdot (\vec{B} + \beta \vec{r}) \right) \quad (\text{B3})$$

where \vec{A} , \vec{E} and \vec{B} are vectors, and α , γ and β are matrices such that $|\vec{A} + \alpha\vec{r}| \leq 1$ etc for any $|\vec{r}| \leq 1$. The correlation functions are

$$E_{13} = \sum_{k,\ell=\pm 1} k \cdot \ell \text{tr} \left[\rho P_{\vec{a}_1}^k \right] \text{tr} \left[\Lambda_E \left(\Lambda_A \left(P_{\vec{a}_1}^k \right) \right) P_{\vec{b}_1}^\ell \right] \\ = (\gamma\alpha\vec{a}_1) \cdot \vec{b}_1 + (\vec{v} \cdot \vec{a}_1) \left[\vec{E} + \gamma\vec{A} \right] \cdot \vec{b}_1 \quad (\text{B4})$$

$$E_{14} = \sum_{k,\ell=\pm 1} k \cdot \ell \text{tr} \left[\rho P_{\vec{a}_1}^k \right] \text{tr} \left[\Lambda_B \left(\Lambda_E \left(\Lambda_A \left(P_{\vec{a}_1}^k \right) \right) \right) P_{\vec{b}_2}^\ell \right] \\ = (\beta\gamma\alpha\vec{a}_1) \cdot \vec{b}_2 + (\vec{v} \cdot \vec{a}_1) \left[\vec{B} + \beta \left(\vec{E} + \gamma\vec{A} \right) \right] \cdot \vec{b}_2 \quad (\text{B5})$$

$$E_{23} = \sum_{k,\ell=\pm 1} k \cdot \ell \text{tr} \left[\Lambda_A(\rho) P_{\vec{a}_2}^k \right] \text{tr} \left[\Lambda_E \left(P_{\vec{a}_2}^k \right) P_{\vec{b}_1}^\ell \right] \\ = (\gamma\vec{a}_2) \cdot \vec{b}_1 + \left[\vec{A} + \alpha\vec{v} \right] \cdot \vec{a}_2 \left(\vec{E} \cdot \vec{b}_1 \right) \quad (\text{B6})$$

$$E_{24} = \sum_{k,\ell=\pm 1} k \cdot \ell \text{tr} \left[\Lambda_A(\rho) P_{\vec{a}_2}^k \right] \text{tr} \left[\Lambda_B \left(\Lambda_E \left(P_{\vec{a}_2}^k \right) \right) P_{\vec{b}_2}^\ell \right] \\ = (\beta\gamma\vec{a}_2) \cdot \vec{b}_2 + \left[\vec{A} + \alpha\vec{v} \right] \cdot \vec{a}_2 \left[\vec{B} + \beta\vec{E} \right] \cdot \vec{b}_2. \quad (\text{B7})$$

Hence, the Bell function is

$$\mathcal{B}_Q = (\gamma\alpha\vec{a}_1 + \gamma\vec{a}_2) \cdot \vec{b}_1 + \beta(\gamma\alpha\vec{a}_1 - \gamma\vec{a}_2) \cdot \vec{b}_2 \\ + (\vec{v} \cdot \vec{a}_1) \left\{ \left[\vec{E} + \gamma\vec{A} \right] \cdot \vec{b}_1 + \left[\vec{B} + \beta \left(\vec{E} + \gamma\vec{A} \right) \right] \cdot \vec{b}_2 \right\} \\ + \left[\vec{A} + \alpha\vec{v} \right] \cdot \vec{a}_2 \left\{ \vec{E} \cdot \vec{b}_1 - \left[\vec{B} + \beta\vec{E} \right] \cdot \vec{b}_2 \right\}. \quad (\text{B8})$$

Substituting the variables $\vec{\xi}_1$ and $\vec{\xi}_2$ defined in the main text one directly recovers Eq. (5) of the main text.

Appendix C: For unitary Λ_A and Λ_B , Tsirelson's bound is achieved only when Λ_E is also unitary

Let ρ be the input density matrix with Bloch vector \vec{v} and let Λ_E CPTP parameterised by matrix γ and shift vector \vec{E} . Let Λ_A and Λ_B be unitary channels represented by the matrices α and β , respectively. The Bell's parameter reads

$$\mathcal{B}_{\text{CPTP}} = (\gamma\alpha\vec{a}_1 + \gamma\vec{a}_2) \cdot \vec{b}_1 + (\beta\gamma\alpha\vec{a}_1 + \beta\gamma\vec{a}_2) \cdot \vec{b}_2 \\ + (\vec{v} \cdot \vec{a}_1) \left(\vec{E} \cdot \vec{b}_1 + \beta\vec{E} \cdot \vec{b}_2 \right) \\ + (\alpha\vec{v} \cdot \vec{a}_2) \left(\vec{E} \cdot \vec{b}_1 - \beta\vec{E} \cdot \vec{b}_2 \right) \quad (\text{C1})$$

which simplifies to

$$\mathcal{B}_{\text{CPTP}} = \vec{\eta}_1 \cdot (\vec{b}_1 + \vec{b}_2') + \vec{\eta}_2 \cdot (\vec{b}_1 - \vec{b}_2'). \quad (\text{C2})$$

after introduction of rotated vectors $\vec{a}'_1 = \alpha\vec{a}_1$, $\vec{v}' = \alpha\vec{v}$ and $\vec{b}'_2 = \beta\vec{b}_2$, and where $\vec{\eta}_1 = (\vec{v}' \cdot \vec{a}'_1)\vec{E} + \gamma\vec{a}'_1$ and $\vec{\eta}_2 = (\vec{v}' \cdot \vec{a}_2)\vec{E} + \gamma\vec{a}_2$. Since vectors $\vec{b}_1 + \vec{b}'_2$ and $\vec{b}_1 - \vec{b}'_2$ are orthogonal and their lengths can be parameterised by a single angle $|\vec{b}_1 + \vec{b}'_2| = 2\sin\theta$ and $|\vec{b}_1 - \vec{b}'_2| = 2\cos\theta$, the Tsirelson's bound can only be achieved if both $\vec{\eta}_1$ and $\vec{\eta}_2$ are unit vectors. Since for all channels satisfying $|\vec{E} + \gamma\vec{r}| < 1$ for all unit vectors \vec{r} , also $|\vec{\eta}_i| < 1$, we are left with studies of channels that output at least one pure state.

All completely positive, trace preserving maps can be reduced to the form (up to unitary conjugation before and after the map, which does not affect the value of the Bell's parameter):

$$\gamma = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\phi & 0 \\ 0 & 0 & \cos\theta\cos\phi \end{bmatrix}, \vec{E} = \begin{bmatrix} 0 \\ 0 \\ \sin\theta\sin\phi \end{bmatrix}, \quad (\text{C3})$$

with $\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$. If $\vec{E} = \vec{0}$, then we must have $\sin\theta = 0$ (or $\sin\phi = 0$) and correspondingly $\cos\theta = \pm 1$ (or $\cos\phi = \pm 1$). Thus, there is only one direction along which γ does not shrink its input vector. Therefore, $\gamma\vec{a}'_1 = \pm\gamma\vec{a}_2$ and the Bell's parameter is restricted by its classical bound as Alice effectively chooses only one setting.

If $|\vec{E}| > 0$, then $|\sin\theta| > 0$ and $|\sin\phi| > 0$. Matrix γ applied on an arbitrary unit vector \vec{r} gives now an ellipsoid of radius *strictly less* than 1. Furthermore, shifting such obtained vectors by \vec{E} has to result in a new vector with $|\vec{E} + \gamma\vec{r}| \leq 1$ in order to guarantee that physical states are mapped to physical states. If instead of shifting by \vec{E} we now shift by $(\vec{v} \cdot \vec{a})\vec{E}$, as in our Bell's parameter, the resulting vectors will be shorter than a unit vector except when $|\vec{v} \cdot \vec{a}'_1| = |\vec{v} \cdot \vec{a}_2| = 1$. In this case, however, the settings of Alice are again the same, along \vec{v} , and therefore the temporal Bell's parameter satisfies the classical bound.

Summing up, the maximal violation can only occur if the channel in-between Alice and Bob is unitary.

Appendix D: Parameter counting

CPTP maps can be written using Kraus operators: $\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$. Extremal qubit maps have Kraus rank 2. In Ref. [34] the operators could be reduced down to the following form $A_i = U S_i V^\dagger$ with U, V unitary and

$$S_1 = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & \sqrt{1-t^2} \\ \sqrt{1-s^2} & 0 \end{pmatrix}, \quad (\text{D1})$$

where $0 \leq s, t \leq 1$. Each unitary map introduces four parameters, so each extremal map has ten parameters. Three such maps, for each time step, gives 30 parameters.